Solutions of the resit exam of WIKR-06

12 July 2018

1a

We have

$$p_X(1) = \mathbb{P}(X = 1) = p_{X,Y,Z}(1,0,0) + p_{X,Y,Z}(1,0,1) + p_{X,Y,Z}(1,1,0) + p_{X,Y,Z}(1,1,1) = 4a,$$

and hence $p_X(0) = 1 - p_X(1) = 1 - 4a$.

Similarly we determine that $p_Y = p_Z = p_X$.

1b

Since X takes values $0, 1$ we have

$$\mathbb{E}X = \sum_x \mathbb{P}(X = x) = \mathbb{P}(X = 1) = 4a.$$

Similarly, we have

$$\begin{aligned}
\mathbb{E}(X|Y = i, Z = j) &= \sum_x x\mathbb{P}(X = x|Y = i, Z = j) = \mathbb{P}(X = 1|Y = i, Z = j) \\
&= \frac{p_{X,Y,Z}(1,i,j)}{p_{X,Y,Z}(0,i,j) + p_{X,Y,Z}(1,i,j)} \\
&= \begin{cases} \frac{1}{2} & \text{if } (i,j) \neq (0,0), \\ \frac{a}{1-6a} & \text{if } (i,j) = (0,0). \end{cases}
\end{aligned}$$

and,

$$\begin{aligned}
\mathbb{E}(X|Y = i) &= \sum_x x\mathbb{P}(X = x|Y = i) = \mathbb{P}(X = 1|Y = i) \\
&= \frac{p_{X,Y,Z}(1,i,0) + p_{X,Y,Z}(1,i,1)}{p_{X,Y,Z}(0,i,0) + p_{X,Y,Z}(0,i,1) + p_{X,Y,Z}(1,i,0) + p_{X,Y,Z}(1,i,1)} \\
&= \begin{cases} \frac{1}{2} & \text{if } i \neq 0, \\ \frac{2a}{1-4a} & \text{if } i = (0,0). \end{cases}
\end{aligned}$$

1c

First note that (by symmetry) of course $\mathbb{E}Y = \mathbb{E}X$, and recall

$$\mathrm{Cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y.$$

Furthermore

$$\mathbb{E}XY = \mathbb{P}(XY = 1) = p_{X,Y,Z}(1,1,0) + p_{X,Y,Z}(1,1,1) = 2a.$$

Hence

$$\mathrm{Cov}(X,Y) = 2a - 16a^2 = 2a(1-8a).$$

1d

Recall from lecture notes that if X, Y are independent then $\mathrm{Cov}(X,Y) = 0$. Hence, by 1c, X, Y must be dependent for $a \notin \{0, \frac{1}{8}\}$.

On the other hand they are independent when $a = 0$ or $a = \frac{1}{8}$. To see this when $a = 0$ note that then $X = Y = 0$ with probability one and hence

$$\begin{aligned}\mathbb{P}(X = x, Y = x) &= \begin{cases} 1 & \text{if } x = y = 0, \\ 0 & \text{otherwise.} \end{cases} \\ &= p_X(x)p_Y(y).\end{aligned}$$

To see it when $a = \frac{1}{8}$, note that in that case, for all $x, y \in \{0,1\}$:

$$\mathbb{P}(X = x, Y = y) = p_{X,Y,Z}(x,y,0) + p_{X,Y,Z}(x,y,1) = \frac{2}{8} = (1/2)^2 = p_X(x)p_Y(y).$$

1e

We first determine the pmf:

$$p_{X+Y+Z}(x) = \begin{cases} 1-7a & \text{if } x = 0, \\ 3a & \text{if } x = 1, \\ 3a & \text{if } x = 2, \\ a & \text{if } x = 3. \end{cases}$$

Thus the cdf is

$$F_{X+Y+Z}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1-7a & \text{if } 0 \le x < 1, \\ 1-4a & \text{if } 1 \le x < 2, \\ 1-a & \text{if } 2 \le x < 3, \\ 1 & \text{if } x \ge 3. \end{cases}$$

2a

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\, dt \overset{t=x^2/2}{=} \int_0^\infty e^{-x^2/2}\sqrt{2}\, dx = \sqrt{2} \cdot \frac{1}{2} \cdot \int_{-\infty}^\infty e^{-x^2/2}\, dx = \sqrt{2} \cdot \frac{1}{2} \cdot \sqrt{2\pi} = \sqrt{\pi}.$$

using the substitution from the hint (giving $dx = (2t)^{-1/2}\, dt$), that $x \mapsto e^{-x^2/2}$ is an even function and the second part of the hint.

2b

Write $Z = X^2$. Clearly $f_Z(x) = 0$ for $x < 0$. Using the hint we have for $x \geq 0$:

$$\begin{aligned} f_Z(x) &= \tfrac{d}{dx}F_X(\sqrt{x}) - \tfrac{d}{dx}F_X(-\sqrt{x}) = \tfrac{1}{2\sqrt{x}} \cdot \tfrac{1}{\sqrt{2\pi}} e^{-x/2} - \tfrac{-1}{2\sqrt{x}} \cdot \tfrac{1}{\sqrt{2\pi}} e^{-x/2} \\ &= \tfrac{x^{-1/2} e^{-x/2}}{\sqrt{\pi} \cdot \sqrt{2}} = \tfrac{x^{1/2-1} e^{-x/2}}{\Gamma(1/2) \cdot 2^{1/2}} \end{aligned}$$

(using a).

2c

Since X_1, X_2 are nonnegative $f_X(x) = 0$ for $x < 0$. For $x \geq 0$, by the convolution formula:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X_1}(s) f_{X_2}(x-s)\, ds \\ &= \int_0^x \frac{s^{\alpha_1 - 1} e^{s/\beta}}{\Gamma(\alpha_1)\beta^{\alpha_1}} \cdot \frac{(x-s)^{\alpha_2-1} e^{(x-s)/\beta}}{\Gamma(\alpha_2)\beta^{\alpha_2}}\, ds \\ &= \frac{e^{x/\beta}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} \int_0^x s^{\alpha_1}(x-s)^{\alpha_2}\, ds \\ &= \frac{e^{x/\beta}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} x^{\alpha_1+\alpha_2-2} \int_0^x (s/x)^{\alpha_1-1}(1-(s/x))^{\alpha_2-1}\, ds \\ &\overset{t=s/x}{=} \frac{e^{x/\beta}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} x^{\alpha_1+\alpha_2-1} \int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\, dt \end{aligned}$$

(Note the substitution $t = s/x$ in the last line gives $ds = x\, dt$.)

2d

Since f_X is a pdf we must have

$$1 = \int_{-\infty}^{\infty} f_X(x)\, dx = \frac{\int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\, dt}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} \cdot \int_0^{\infty} x^{\alpha_1+\alpha_2-1} e^{x/\beta}\, dx.$$

On the other hand, the density of the Gamma$(\alpha_1 + \alpha_2, \beta)$-distribution also integrates to one. I.e.

$$1 = \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \cdot \int_0^{\infty} x^{\alpha_1+\alpha_2-1} e^{x/\beta}\, dx.$$

Combining these those equalities gives

$$\int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\, dt = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}}{\int_0^{\infty} x^{\alpha_1+\alpha_2-1} e^{x/\beta}\, dx} \cdot \frac{\int_0^{\infty} x^{\alpha_1+\alpha_2-1} e^{x/\beta}\, dx}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}.$$

(Alternatively, you could have used the substitution $t = x/\beta$ to compute that $\int_0^{\infty} x^{\alpha_1+\alpha_2-1} e^{x/\beta}\, dx = \beta^{\alpha_1+\alpha_1}\Gamma(\alpha_1+\alpha_2)$.)

2e

Write $Y_i := X_i^2$. Then $Y_1, \ldots, Y_n$ are i.i.d. Gamma$(1/2, 2)$-distributed by part b. Combining 2c and 2d, we know that if $Z_1 \sim$ Gamma(α_1, β) and $Z_2 \sim$ Gamma(α_2, β) are independent, then $Z_1 + Z_2 \sim$ Gamma$(\alpha_1 + \alpha_2, \beta)$. Repeated applications of this observation show $Y_1 + \cdots + Y_n \sim$ Gamma$(n/2, 2)$.

2f

This is the Exp(2)-distribution. That is, the exponential distribution with parameter $\beta = 2$.

3a

Let $C_i := \{\text{flea is on cat at time } i\}$. We have

$$\begin{aligned}\mathbb{P}(C_{n+1}) &= \mathbb{P}(C_{n+1}|C_n^c)\cdot\mathbb{P}(C_n^c) + \mathbb{P}(C_{n+1}|C_n)\cdot\mathbb{P}(C_n)\\ &= \mathbb{P}(\text{jump in the } (n+1)\text{-st second}|C_n^c)\cdot\mathbb{P}(C_n^c) + \mathbb{P}(\text{stay put in the } (n+1)\text{-st second}|C_n)\cdot\mathbb{P}(C_n)\\ &= p(1-p_n) + (1-p)p_n.\end{aligned}$$

3b

Note the recursion from 3a can be rewritten as

$$p_{n+1} = p + (1-2p)p_n.$$

Repeated applications of this gives

$$\begin{aligned}p_{n+1} &= p + (1-2p)(p + (1-2p)p_{n-1})\\ &= p + (1-2p)p + (1-2p)^2 p_{n-1}\\ &= p + (1-2p)p + (1-2p)^2 p + (1-2p)^3 p_{n-2}\\ &\;\;\vdots\\ &= p + (1-2p)p + \cdots + (1-2p)^n p + (1-2p)^{n+1} p_0\\ &= p\left(1 + (1-2p) + \cdots + (1-2p)^n\right) + (1-2p)^{n+1} p_0\\ &= p\left(\frac{1-(1-2p)^{n+1}}{1-(1-2p)}\right) + (1-2p)^{n+1} p_0\\ &= \frac{1}{2}\left(1-(1-2p)^{n+1}\right) + (1-2p)^{n+1} p_0.\end{aligned}$$

3c

Suppose that the flea is initially on the cat. Then p_n is exactly the probability that an even number of jumps occur in the first n seconds. Which also equals $\mathbb{P}(X_n \text{ is even })$. Hence, using 3b and that $p_0 = 1$ if the flea starts on the cat:

$$\mathbb{P}(X_n \text{ is even }) = \frac{1}{2}\left(1-(1-2p)^n\right) + (1-2p)^n = \frac{1}{2} + \frac{1}{2}(1-2p)^n.$$

So in particular $\mathbb{P}(X_n \text{ is even }) \overset{n\to\infty}{\longrightarrow} \frac{1}{2}$.

(Important : we are using that $p \neq 0, 1$ so that $|1-2p| < 1$. Otherwise it is of course false!)

3d

Now we get the recursion

$$p_{n+1} = (1-p_{\text{cat}})p_n + p_{\text{dog}}(1-p_n) = p_{\text{dog}} + (1-p_{\text{cat}}-p_{\text{dog}})p_n.$$

Repeated applications of this, and continuing as in 3b we get

$$p_{n+1} = p_{\mathrm{dog}} \left(\frac{1 - (1 - p_{\mathrm{cat}} - p_{\mathrm{dog}})^{n+1}}{p_{\mathrm{cat}} + p_{\mathrm{dog}}} \right) + p_0 (1 - p_{\mathrm{cat}} - p_{\mathrm{dog}})^{n+1}.$$

Since $0 < p_{\mathrm{cat}}, p_{\mathrm{dog}} < 1$ we have $|1 - p_{\mathrm{cat}} - p_{\mathrm{dog}}| < 1$ so that $(1 - p_{\mathrm{cat}} - p_{\mathrm{dog}})^n \overset{n\to\infty}{\longrightarrow} 0$ and hence

$$p_n \overset{n\to\infty}{\longrightarrow} \frac{p_{\mathrm{dog}}}{p_{\mathrm{cat}} + p_{\mathrm{dog}}},$$

no matter the value of p_0.